

# Decompositions of complete multipartite graphs via generalized graceful labelings

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## Abstract

We prove the existence of infinite classes of cyclic  $\Gamma$ -decompositions of the complete multipartite graph,  $\Gamma$  being a caterpillar, a hairy cycle or a cycle. All the results are obtained by the construction of  $d$ -divisible  $\alpha$ -labelings of  $\Gamma$ , introduced in [15] as a generalization of classical  $\alpha$ -labelings, whose existence implies that one of graph decompositions.

**Keywords:** graph decomposition; graceful labeling;  $d$ -divisible graceful labeling.

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## 1 Introduction

As usual, we denote by  $K_v$  and  $K_{m \times n}$  the *complete graph on  $v$  vertices* and the *complete  $m$ -partite graph with parts of size  $n$* , respectively. For any graph  $\Gamma$  we write  $V(\Gamma)$  for the set of its vertices and  $E(\Gamma)$  for the set of its edges. If  $|E(\Gamma)| = e$ , we say that  $\Gamma$  has *size  $e$* .

Given a subgraph  $\Gamma$  of a graph  $K$ , a  $\Gamma$ -*decomposition of  $K$*  is a set of graphs isomorphic to  $\Gamma$ , called *blocks*, whose edges partition the edge-set of  $K$ . Such a decomposition is said to be *cyclic* when it is invariant under a cyclic permutation of all the vertices of  $K$ . A  $\Gamma$ -decomposition of  $K_v$  is also called a  $\Gamma$ -*system of order  $v$* . For a survey on graph decompositions see [6].

The concept of a *graceful labeling* of  $\Gamma$ , introduced by A. Rosa [16], is a useful tool working indirectly to determine infinite classes of  $\Gamma$ -systems. Rosa proved that if a graph  $\Gamma$  of size  $e$  admits a graceful labeling then there exists a cyclic  $\Gamma$ -system of order  $2e + 1$  and if  $\Gamma$  admits an  $\alpha$ -labeling then there exists a cyclic  $\Gamma$ -system of order  $2en + 1$  for any positive integer  $n$ . Also, labeled graphs are models for a broad range of applications, see for instance [4, 5]. For a very rich survey on graceful labelings we refer to [11].

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Here we recall the basic definition. A *graceful labeling* of a graph  $\Gamma$  of size  $e$  is an injective function  $f : V(\Gamma) \rightarrow \{0, 1, 2, \dots, e\}$  such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, \dots, e\}.$$

In the case where  $\Gamma$  is bipartite and  $f$  has the additional property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one, one says that  $f$  is an  $\alpha$ -labeling.

Many variations of graceful labelings have been considered over the years. In particular Gnana Jothi [12] defines an *odd graceful labeling* of a graph  $\Gamma$  of size  $e$  as an injective function  $f : V(\Gamma) \rightarrow \{0, 1, 2, \dots, 2e - 1\}$  such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 3, 5, \dots, 2e - 1\}.$$

In a recent paper, see [15], the second author have introduced the following new definition which is, at the same time, a generalization of the concepts of a graceful labeling (when  $d = 1$ ) and of an odd graceful labeling (when  $d = e$ ).

**Definition 1.1.** Let  $\Gamma$  be a graph of size  $e = d \cdot m$ . A  $d$ -divisible graceful labeling of  $\Gamma$  is an injective function  $f : V(\Gamma) \rightarrow \{0, 1, 2, \dots, d(m + 1) - 1\}$  such that

$$\begin{aligned} \{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} &= \{1, 2, 3, \dots, d(m + 1) - 1\} \\ &\quad \cup \{m + 1, 2(m + 1), \dots, (d - 1)(m + 1)\}. \end{aligned}$$

Namely the set  $\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\}$  can be divided into  $d$  parts  $P^0, P^1, \dots, P^{d-1}$  where  $P^i := \{(m + 1)i + 1, (m + 1)i + 2, \dots, (m + 1)i + m\}$  for any  $i = 0, 1, \dots, d - 1$ .

By saying that  $d$  is *admissible* we will mean that it is a divisor of  $e$  and so it makes sense to investigate the existence of a  $d$ -divisible graceful labeling of  $\Gamma$ . Also  $\alpha$ -labelings can be generalized in a similar way.

**Definition 1.2.** A  $d$ -divisible  $\alpha$ -labeling of a bipartite graph  $\Gamma$  is a  $d$ -divisible graceful labeling of  $\Gamma$  having the property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one.

Results on the existence of  $d$ -divisible  $\alpha$ -labelings can be found in [14, 15]. In particular, using the notion of a  $(v, d, \Gamma, 1)$ -difference family introduced in [7], in [15] it is proved that the existence of a  $d$ -divisible ( $\alpha$ -)labeling of a graph  $\Gamma$  implies the existence of cyclic  $\Gamma$ -decompositions. In fact:

**Theorem 1.3.** If there exists a  $d$ -divisible graceful labeling of a graph  $\Gamma$  of size  $e$  then there exists a cyclic  $\Gamma$ -decomposition of  $K_{(\frac{e}{d}+1) \times 2d}$ .

**Theorem 1.4.** *If there exists a  $d$ -divisible  $\alpha$ -labeling of a graph  $\Gamma$  of size  $e$  then there exists a cyclic  $\Gamma$ -decomposition of  $K_{(\frac{e}{d}+1)\times 2dn}$  for any positive integer  $n$ .*

In this paper we deal with the existence of  $d$ -divisible  $\alpha$ -labelings of caterpillars, hairy cycles and cycles. Often, in the literature, the problem of determining the existence of a labeling of a given graph is solved by an “ad hoc” construction directly related to the case considered. Here, in Section 2, we will construct a  $d$ -divisible  $\alpha$ -labeling of a caterpillar for any admissible value of  $d$ . Also, we define a new kind of labeling for a graph  $\Gamma$ , we show that if  $\Gamma$  is a caterpillar such a labeling always exists and we call it *standard  $\alpha_S$ -labeling* of  $\Gamma$ . In Section 3, starting from a standard  $\alpha_S$ -labeling of a suitable caterpillar we will construct a  $d$ -divisible  $\alpha$ -labeling of a hairy cycle or a cycle. In detail, we will show that bipartite hairy cycles admit an odd  $\alpha$ -labeling and, when all the vertices of the cycle have the same degree, they admit a  $d$ -divisible  $\alpha$ -labeling for any admissible value of  $d$ . About cycles we will show that, for any positive integer  $k$ ,  $C_{4k}$  admits a  $d$ -divisible  $\alpha$ -labeling for any admissible value of  $d$ .

We point out that in the cases of caterpillars and cycles, if we consider the  $d$ -divisible  $\alpha$ -labelings here presented with  $d = 1$  we can obtain again the  $\alpha$ -labelings presented by Rosa in [16].

The existence of these  $d$ -divisible  $\alpha$ -labelings allows us to obtain new infinite classes of decompositions of the complete multipartite graph. Here we obtain a cyclic  $\Gamma$ -decomposition of  $K_{m\times n}$  for infinite pairs  $(m, n)$  and for any caterpillar or hairy cycle  $\Gamma$ .

The problem of decomposing  $K_{m\times n}$  into cycles has been taken in part; for instance when the cycle length is a prime [13], twice a prime [17] or “small” [9]. In [8, 2, 3] the authors gave necessary and sufficient condition for the existence of cycle decompositions of  $K_{3\times n}$ ,  $K_{4\times n}$  and  $K_{5\times n}$ , respectively. Here we obtain cyclic  $C_{4k}$ -decompositions of  $K_{m\times n}$  for any positive integer  $k$  and for infinite pairs  $(m, n)$ .

## 2 Caterpillars

**Definition 2.1.** *A caterpillar  $\Gamma$  is a tree, namely a graph without cycles, with the property that the removal of its vertices of degree one leaves a path  $P$ , called the path associated to  $\Gamma$ .*

By the definition, any path is a caterpillar too. In [16], Rosa proved that any caterpillar admits an  $\alpha$ -labeling. In this section we will show, more in general, that any caterpillar admits a  $d$ -divisible  $\alpha$ -labeling for any admissible value of  $d$ .

First of all we introduce the following notation for the vertices of a caterpillar to emphasize that we will see it as a bipartite graph.

Let  $\Gamma$  be a caterpillar and let  $P$  be the path associated to  $\Gamma$ . If the length of  $P$  is even, set  $P = [x_1, y_1, x_2, y_2, \dots, x_t, y_t]$ . Let  $n_i$  be the number of pendant edges through the vertex  $x_i$  and  $m_i$  be the number of pendant edges through the vertex  $y_i$  for  $i = 1, \dots, t$ . Also we call  $x_i^1, x_i^2, \dots, x_i^{n_i}$  the vertices of the edges pendant through the vertex  $x_i$  and  $y_i^1, y_i^2, \dots, y_i^{m_i}$  the vertices of the edges pendant through the vertex  $y_i$  for any  $i = 1, \dots, t$ ; see Figure 1. Hence the two bipartite sets are

$$A = \{x_1, y_1^1, y_1^2, \dots, y_1^{m_1}, x_2, y_2^1, y_2^2, \dots, y_2^{m_2}, \dots, x_t, y_t^1, y_t^2, \dots, y_t^{m_t}\} \quad (1)$$

$$B = \{x_1^1, x_1^2, \dots, x_1^{n_1}, y_1, x_2^1, x_2^2, \dots, x_2^{n_2}, y_2, \dots, x_t^1, x_t^2, \dots, x_t^{n_t}, y_t\} \quad (2)$$

and we will denote such a caterpillar by  $C[n_1, m_1, \dots, n_t, m_t]$ .

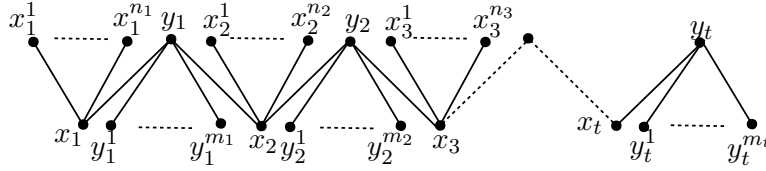


Figure 1: The caterpillar  $C[n_1, m_1, \dots, n_t, m_t]$

Notice that the elements of  $E(\Gamma)$  can be taken in a natural order from  $[x_1, x_1^1]$  up to  $[y_t, y_t^{m_t}]$ , that is we can write

$$E(\Gamma) = \{[x_1, x_1^1], \dots, [x_1, x_1^{n_1}], [x_1, y_1], \dots, [y_1, y_1^{m_1}], [y_1, x_2], \dots, [x_t, y_t], [y_t, y_t^1], \dots, [y_t, y_t^{m_t}]\}. \quad (3)$$

Obviously if  $n_1 = 0$ , (3) becomes

$$E(\Gamma) = \{[x_1, y_1], \dots, [y_1, y_1^{m_1}], [y_1, x_2], \dots, [x_t, y_t], [y_t, y_t^1], \dots, [y_t, y_t^{m_t}]\}.$$

Analogously, if  $m_t = 0$ , we have

$$E(\Gamma) = \{[x_1, x_1^1], \dots, [x_1, x_1^{n_1}], [x_1, y_1], \dots, [y_1, y_1^{m_1}], [y_1, x_2], \dots, [x_t, y_t]\}.$$

In what follows, by *consecutive*, we will mean two vertices of  $A$  or of  $B$  or two edges of  $E(\Gamma)$  consecutive in the order assumed in (1), (2) or (3), respectively.

If the length of  $P$  is odd, we can use analogous notations.

Now we introduce a new concept, which will be very useful in the following, that generalizes that of a  $d$ -divisible  $\alpha$ -labeling.

**Definition 2.2.** Let  $\Gamma$  be a bipartite graph of size  $e$  with parts  $A$  and  $B$ . Let  $S$  be a set of  $e$  positive integers. We call  $\alpha_S$ -labeling of  $\Gamma$  an injective function  $f : V(\Gamma) \rightarrow \{0, 1, 2, \dots, \max_{s \in S} s\}$  such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = S \quad \text{and} \quad \max_A f < \min_B f.$$

**Theorem 2.3.** *Any caterpillar  $\Gamma$  of size  $e$  admits an  $\alpha_S$ -labeling for any set  $S$  of  $e$  positive integers.*

Proof. Let  $\Gamma$  be a caterpillar with parts  $A$  and  $B$  defined as in (1) and (2), respectively. We label the edges of  $\Gamma$  taken in the same order of (3), with the elements of  $S$  in ascending order. Let  $f : V(\Gamma) \rightarrow \{0, 1, 2, \dots, \max_{s \in S} s\}$  be the function defined so that, given  $x \in A$  and  $y \in B$ , the label of the edge  $[x, y]$  is  $f(y) - f(x)$  and with  $f(y_t^{m_t}) = 0$  ( $f(x_t) = 0$  if  $m_t = 0$ ). Note that if we consider the elements of  $A$  in the same order of (1) their labels are in descending order and if we consider the elements of  $B$  in the same order of (2) their labels are in ascending order. So, it is easy to see that  $f$  is an injective function and that  $\max_A f < \min_B f$ . Hence  $f$  is an  $\alpha_S$ -labeling of  $\Gamma$ .  $\square$

**Definition 2.4.** *We will call the function  $f$  constructed in the proof of Theorem 2.3 the standard  $\alpha_S$ -labeling of  $\Gamma$ .*

Thanks to Theorem 2.3 we can completely solve the problem of the existence of  $d$ -divisible  $\alpha$ -labelings of caterpillars.

**Corollary 2.5.** *Any caterpillar admits a  $d$ -divisible  $\alpha$ -labeling for any admissible value of  $d$ .*

Proof. Let  $\Gamma$  be a caterpillar with  $e = d \cdot m$  edges. Let  $f$  be the standard  $\alpha_S$ -labeling of  $\Gamma$  where  $S = \{1, 2, \dots, d(m+1) - 1\} \setminus \{m+1, 2(m+2), \dots, (d-1)(m+1)\}$ . It is easy to see that  $f$  is a  $d$ -divisible  $\alpha$ -labeling of  $\Gamma$ .  $\square$

As an immediate consequence of Theorem 1.4 and Corollary 2.5 we have:

**Theorem 2.6.** *Let  $\Gamma$  be a caterpillar with  $e$  edges. There exists a cyclic  $\Gamma$ -decomposition of  $K_{(\frac{e}{d}+1) \times 2dn}$  for any divisor  $d$  of  $e$  and any positive integer  $n$ .*

**Remark 2.7.** *We point out that if  $d = 1$  the  $d$ -divisible  $\alpha$ -labeling of the proof of Corollary 2.5 is nothing but the  $\alpha$ -labeling obtained by Rosa. Also, if the caterpillar is indeed a path, we find again the  $d$ -divisible  $\alpha$ -labeling constructed in [15].*

**Example 2.8.** *We consider a caterpillar  $\Gamma$  with 12 edges. In Figure 2 we show all the possible  $d$ -graceful  $\alpha$ -labelings of  $\Gamma$ , other than the classical one, obtained following the proof of Corollary 2.5.*

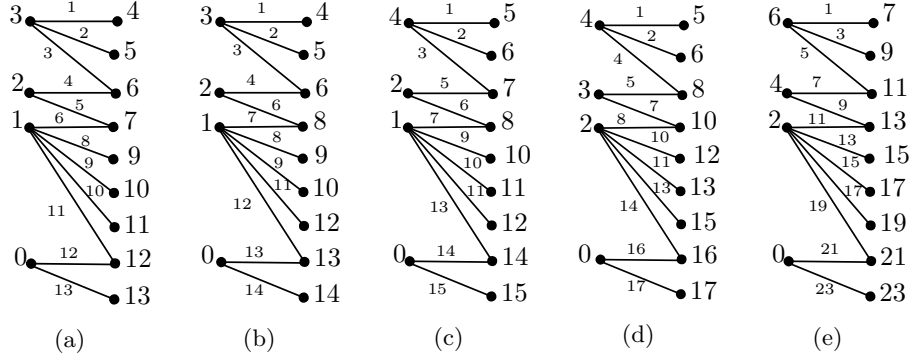


Figure 2:  $d$ -divisible  $\alpha$ -labelings of a caterpillar

**Definition 2.9.** Let  $f$  be an  $\alpha_S$ -labeling of a graph  $\Gamma$  and let  $X \subseteq V(\Gamma)$ . We will call missing vertex label in  $f(X)$  (mv-label, for short) each element of  $\{n \in \mathbb{N} \mid \min_X f \leq n \leq \max_X f\} \setminus f(X)$ .

We point out that following the construction of the  $\alpha_S$ -labeling provided in Theorem 2.3 each missing integer in  $S$  causes a corresponding mv-label in  $f(A) \cup f(B)$ . For instance, look at Figure 2(d). Now  $S = \{1, 2, \dots, 17\} \setminus \{3, 6, 9, 12, 15\}$  and the missing integers  $\{3, 6, 9, 12, 15\}$  cause the mv-labels  $\{7, 9, 11, 14, 1\}$ , respectively.

### 3 Hairy cycles and cycles from caterpillars

**Definition 3.1.** A unicyclic graph  $\Gamma$ , other than a cycle, is called a hairy cycle if the deletion of any edge in the cycle of  $\Gamma$  results in a caterpillar.

So, both cycles and hairy cycles can be always seen as suitable caterpillars  $\Gamma$  with one extra edge: the one connecting the ending vertices of the path associated to  $\Gamma$ .

Following this line and keeping in mind the construction of Rosa (see [16]), in [1] Barrientos proves that all hairy cycles are graceful and that when the graph is bipartite, namely when the cycle has even length, the labeling is an  $\alpha$ -labeling.

Following the same line and using the results of Section 2, in the sequel we will give a method to construct  $d$ -divisible  $\alpha$ -labelings of hairy cycles and cycles. Thus, from now on we will draw hairy cycles and cycles as in Figure 3(b) but instead in the classical way of Figure 3(a) and by  $HC(n_1, m_1, \dots, n_t, m_t)$ , with  $t \geq 2$ , we will denote the hairy cycle obtainable from the caterpillar  $C[n_1, m_1, \dots, n_t, m_t]$  by adding the edge  $[x_1, y_t]$ . If the cycle  $(x_1, y_1, \dots, x_{t-1}, y_{t-1}, x_t)$ , with  $t \geq 2$ , had odd length, we would obviously refer for the constructed hairy cycle with  $HC(n_1, m_1, \dots, n_{t-1}, m_{t-1}, n_t)$ .

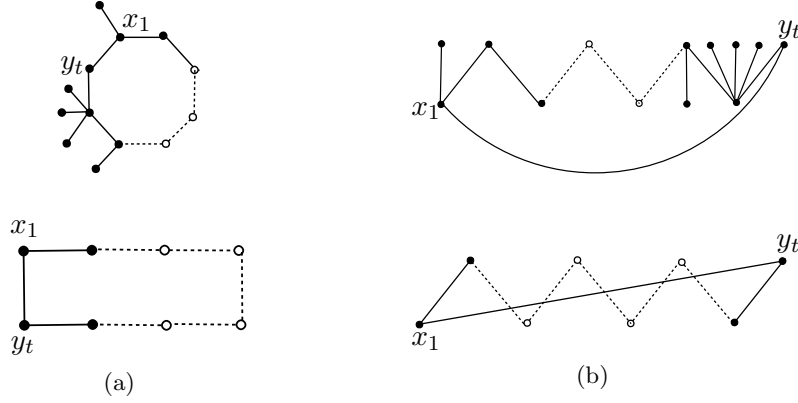


Figure 3: An hairy cycle and a cycle

**Remark 3.2.** Consider the graphs of Figure 3(b) and let  $e$  be their size. Removing the edge  $[x_1, y_t]$  we obtain, in both cases, a caterpillar  $\Gamma$  of size  $e - 1$ . Let  $S = \{a_1, a_2, \dots, a_e\}$  be a set of  $e$  integers. Set  $S' = \{a_1, a_2, \dots, a_e\} \setminus \{a_c\}$ ,  $1 \leq c \leq e$ , and consider an  $\alpha_{S'}$ -labeling of  $\Gamma$ , say  $f$ . If it results in  $f(y_t) - f(x_1) = a_c$ , in a natural way we can extend  $f$  to an  $\alpha_S$ -labeling of the original graph, labeling the edge  $[x_1, y_t]$  by  $a_c$ .

Let the elements of  $S$  be taken in ascending order. Let  $S'$  be defined as in previous remark and  $f$  be the standard  $\alpha_{S'}$ -labeling of  $\Gamma$ . Obviously, the positions of the mv-labels affect the value of  $f(y_t) - f(x_1)$ . In particular, notice that if  $a_{c-1}$  and  $a_{c+1}$  label two consecutive edges through a vertex of  $B$  (or  $A$ ), the deletion of  $a_c$  causes an extra mv-label in  $f(A)$  (or  $f(B)$ ). So, given an  $\alpha_S$ -labeling  $f$  of a caterpillar  $\Gamma$  we can try to lightly modify  $f$ , if it is the case, changing the position of some mv-labels in order to obtain another function  $g$  which results in still an  $\alpha_S$ -labeling of  $\Gamma$  with  $g(y_t) - g(x_1) \neq f(y_t) - f(x_1)$ . Here we are giving some few ways to obtain functions as  $g$ .

In the sequel  $\Gamma$  is a caterpillar (see Figure 1),  $S$  is an arbitrary set of positive integers of size  $e = |E(\Gamma)|$ ,  $f$  is the standard  $\alpha_S$ -labeling of  $\Gamma$ . If  $x$  and  $y$  are consecutive vertices of  $A$  or of  $B$ ,  $f(x) + h = f(y)$  means that there are  $h - 1$  mv-labels between  $f(x)$  and  $f(y)$ .

• **[O<sub>1</sub>]** gives  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1$ .

If there exists  $s \in \{1, 2, \dots, t-1\}$  such that the following conditions are true:

(1)  $f(y_s) + 1 = f(x_{s+1}^1)$ , (2)  $f(y_s) = f(x_s^{n_s}) + h$ , with  $h \geq 2$ ,

then, the function  $g$  defined as follows:

$$g(x_{s+1}^1) = f(x_{s+1}^1) - 1$$

$$g(z) = f(z) \quad \forall z \in \{y_t^{m_t}, \dots, x_{s+1}\} \cup \{x_{s+1}^2, \dots, y_t\} = H_1$$

$$g(z) = f(z) + 1 \quad \forall z \in \{y_s^{m_s}, \dots, x_1\} \cup \{x_1^1, \dots, y_s\} = H_2$$

results in again an  $\alpha_S$ -labeling of  $\Gamma$ .

In fact,

1.  $S$  results in the same because the labels of the edges from  $[x_1, x_1^1]$  up to  $[y_s^{m_s}, y_s]$  as well as those from  $[x_{s+1}, x_{s+1}^2]$  up to  $[y_t^{m_t}, y_t]$  are the same while those of  $[x_{s+1}, y_s]$  and  $[x_{s+1}, x_{s+1}^1]$  commute.
2.  $g$  is injective because  $g(z_1) \neq g(z_2)$  for  $z_1, z_2 \in H_1 \cup H_2$  with  $z_1 \neq z_2$ . If, by contradiction,  $g(y_s) = g(x_{s+1}^1)$  then  $f(y_s) + 1 = f(x_{s+1}^1) - 1$  that is  $f(x_{s+1}^1) = f(y_s) + 2$ , excluded by (1). If, again by contradiction,  $g(x_{s+1}^1) = g(x_s^{n_s})$  then  $f(x_{s+1}^1) - 1 = f(x_s^{n_s}) + 1$  that is  $f(y_s) = f(x_s^{n_s}) + 1$ , excluded by (2).
3.  $\max_A g = \max_A f + 1 < \min_B f + 1 = \min_B g$ , so  $g$  results in an  $\alpha_S$ -labeling of  $\Gamma$ .

• **[O<sub>2</sub>]** gives  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2$ .

If there exists  $s \in \{1, 2, \dots, t-1\}$  such that the following conditions are true:

(1)  $f(y_s) + 2 = f(x_{s+1}^1)$ , (2)  $f(y_s) = f(x_s^{n_s}) + h$ , with  $h \geq 3$ ,

then, the function  $g$  defined as follows:

$$g(x_{s+1}^1) = f(x_{s+1}^1) - 2$$

$$g(z) = f(z) \quad \forall z \in \{y_t^{m_t}, \dots, x_{s+1}\} \cup \{x_{s+1}^2, \dots, y_t\} = H_1$$

$$g(z) = f(z) + 2 \quad \forall z \in \{y_s^{m_s}, \dots, x_1\} \cup \{x_1^1, \dots, y_s\} = H_2$$

results in again an  $\alpha_S$ -labeling of  $\Gamma$ .

In fact,

1.  $S$  results in the same because the labels of the edges from  $[x_1, x_1^1]$  up to  $[y_s^{m_s}, y_s]$  as well as those from  $[x_{s+1}, x_{s+1}^2]$  up to  $[y_t^{m_t}, y_t]$  are the same while those of  $[x_{s+1}, y_s]$  and  $[x_{s+1}, x_{s+1}^1]$  commute.
2.  $g$  is injective because  $g(z_1) \neq g(z_2)$  for  $z_1, z_2 \in H_1 \cup H_2$  with  $z_1 \neq z_2$ . If, by contradiction,  $g(y_s) = g(x_{s+1}^1)$  then  $f(y_s) + 2 = f(x_{s+1}^1) - 2$  that is  $f(x_{s+1}^1) = f(y_s) + 4$ , excluded by (1). If, again by contradiction,  $g(x_{s+1}^1) = g(x_s^{n_s})$  then  $f(x_{s+1}^1) - 2 = f(x_s^{n_s}) + 2$  that is  $f(y_s) = f(x_s^{n_s}) + 2$ , excluded by (2).
3.  $\max_A g = \max_A f + 2 < \min_B f + 2 = \min_B g$ , so  $g$  results in an  $\alpha_S$ -labeling of  $\Gamma$ .

• **[O<sub>3</sub>]** gives  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2$ .

If there exists  $s \in \{1, 2, \dots, t-1\}$  such that the following conditions are true:

(1)  $f(y_s) + 2 = f(x_{s+1}^1) + 1 = f(x_{s+1}^2)$ , (2)  $f(y_s) = f(x_s^{n_s}) + h$ , with  $h \geq 3$ ,

then, the function  $g$  defined as follows:

$$g(x_{s+1}^2) = f(x_{s+1}^2) - 2, \quad f(x_{s+1}^1) = g(x_{s+1}^1)$$

$$g(z) = f(z) \quad \forall z \in \{y_t^{m_t}, \dots, x_{s+1}\} \cup \{x_{s+1}^3, \dots, y_t\} = H_1$$

$$g(z) = f(z) + 2 \quad \forall z \in \{y_s^{m_s}, \dots, x_1\} \cup \{x_1^1, \dots, y_s\} = H_2$$

results in again an  $\alpha_S$ -labeling of  $\Gamma$ .

In fact,

1.  $S$  results in the same because the labels of the edges from  $[x_1, x_1^1]$  up to  $[y_s^{m_s}, y_s]$ , those of the edges from  $[x_{s+1}, x_{s+1}^3]$  up to  $[y_t^{m_t}, y_t]$  as well as the label of  $[x_{s+1}, x_{s+1}^1]$  are the same while those of  $[x_{s+1}, y_s]$  and  $[x_{s+1}, x_{s+1}^2]$  commute.
2.  $g$  is injective because  $g(z_1) \neq g(z_2)$  for  $z_1, z_2 \in H_1 \cup H_2$  with  $z_1 \neq z_2$ . If,



by contradiction,  $g(y_s) = g(x_{s+1}^1)$  then  $f(y_s) + 2 = f(x_{s+1}^1)$ , excluded by (1).  
 If, by contradiction,  $g(y_s) = g(x_{s+1}^2)$  then  $f(y_s) + 2 = f(x_{s+1}^2) - 2 = f(y_s)$ , excluded.  
 If, again by contradiction,  $g(x_{s+1}^2) = g(x_s^{n_s})$  then  $f(x_{s+1}^2) - 2 = f(x_s^{n_s}) + 2$  that is  $f(y_s) = f(x_s^{n_s}) + 2$ , excluded by (2).  
 3.  $\max_A g = \max_A f + 2 < \min_B f + 2 = \min_B g$ , so  $g$  results in an  $\alpha$ -labeling of  $\Gamma$ .

• **[O<sub>4</sub>]** gives  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - h$  where  $h = f(y_t) - f(x_t^{n_t})$ .  
*If the following conditions are true: (1)  $n_t \neq 0$ , (2)  $m_t = 0$ , then, the function  $g$  defined as follows:*  
 $g(z) = f(z) \quad \forall z \in V(\Gamma) \setminus \{y_t, x_t^{n_t}\}$ ,  $g(y_t) = f(x_t^{n_t})$  and  $g(x_t^{n_t}) = f(y_t)$   
*results in again an  $\alpha_S$ -labeling of  $\Gamma$ .*

In fact,

1.  $S$  results in the same because the labels of the edges are the same with the exception of those of  $[x_t, y_t]$  and  $[x_t, x_t^{n_t}]$  that commute.
2.  $g$  is injective because  $g(z) = f(z)$  for each  $z \in V(\Gamma)$  with the exception of  $y_t$  and  $x_t^{n_t}$ , whose images are swapped.
3.  $\max_A g = \max_A f < \min_B f = \min_B g$ , so  $g$  results in an  $\alpha_S$ -labeling of  $\Gamma$ .

• **[O<sub>5</sub>]** gives  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1$ .  
*If there exists  $s \in \{1, 2, \dots, t-1\}$  such that the following conditions are true:*  
 (1)  $f(y_s) = f(x_{s+1}^1) - 2$ , (2)  $f(y_s) = f(x_s^{n_s}) + 1$ ,  
*then, the function  $g$  defined as follows:*  
 $g(x_s^{n_s}) = f(x_s^{n_s}) + 2$ ,  
 $g(z) = f(z) \quad \forall z \in \{y_t^{m_t}, \dots, y_s^1\} \cup \{y_s, \dots, y_t\} = H_1$   
 $g(z) = f(z) + 1 \quad \forall z \in \{x_s, \dots, x_1\} \cup \{x_1^1, \dots, x_s^{n_s-1}\} = H_2$   
*results in again an  $\alpha_S$ -labeling of  $\Gamma$ .*

In fact,

1.  $S$  results in the same because the labels of the edges from  $[x_1, x_1^1]$  up to  $[x_s^{n_s-1}, x_s]$  and those of the edges from  $[y_s, y_s^1]$  up to  $[y_t^{m_t}, y_t]$  are the same while those of  $[x_s, x_s^{n_s}]$  and  $[x_s, y_s]$  commute.
2.  $g$  is injective because  $g(z_1) \neq g(z_2)$  for  $z_1, z_2 \in H_1 \cup H_2$  with  $z_1 \neq z_2$ . If, by contradiction,  $g(x_s^{n_s-1}) = g(x_s^{n_s})$  then  $f(x_s^{n_s-1}) + 1 = f(x_s^{n_s}) + 2$ , so  $f(x_s^{n_s-1}) = f(x_s^{n_s}) + 1 = f(y_s)$  (by (2)), excluded as  $f$  is injective. Also, if, by contradiction,  $g(x_s^{n_s}) = g(x_{s+1}^1)$  then  $f(x_s^{n_s}) + 2 = f(x_{s+1}^1)$  so  $f(y_s) + 1 = f(y_s) + 2$ , by (1) and (2), which is obviously impossible.
3.  $\max_A g = \max_A f + 1 < \min_B f + 1 = \min_B g$ , so  $g$  results in an  $\alpha_S$ -labeling of  $\Gamma$ .

• **[O<sub>5</sub>]<sub>4</sub>]** gives  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1$ .  
*If there exists  $s \in \{1, 2, \dots, t-1\}$  such that the following conditions are true:*  
 (1)  $\exists j \in \{1, 2, \dots, m_s\}$  such that  $f(y_s^j) = f(x_s) - 4$ , (2)  $f(y_s) = f(x_s^{n_s}) + 1$ ,  
 (3)  $f(x_{s+1}^i) = f(y_s) + i$ ,  $i = 1, 2, 3, 4$ , and  $f(x_{s+1}^5) = f(y_s) + 6$ ,  
*then, the function  $g$  defined as follows:*

$g(x_s^{n_s}) = f(y_s) + 5, \quad g(y_s^j) = f(x_s)$   
 $g(z) = f(z) \quad \forall z \in (\{y_t^{m_t}, \dots, y_s^1\} \setminus \{y_s^j\}) \cup \{y_s, \dots, y_t\} = H_1$   
 $g(z) = f(z) + 1 \quad \forall z \in \{x_s, \dots, x_1\} \cup \{x_1^1, \dots, x_s^{n_s-1}\} = H_2$   
*results in again an  $\alpha_S$ -labeling of  $\Gamma$ .*

In fact,

1.  $S$  results in the same because the labels of the edges from  $[x_1, x_1^1]$  up to  $[x_s^{n_s-1}, x_s]$  and those of the edges from  $[y_s, y_s^1]$  up to  $[y_t^{m_t}, y_t]$ , excepted  $[y_s, y_s^j]$ , are the same while those of  $[x_s, x_s^{n_s}]$ ,  $[x_s, y_s]$  and  $[y_s, y_s^j]$  cyclically permute.
2.  $g$  is injective because  $g(z_1) \neq g(z_2)$  for  $z_1, z_2 \in H_1 \cup H_2$  with  $z_1 \neq z_2$ . If, by contradiction, there was an  $i$  such that  $g(x_s^{n_s}) = g(x_{s+1}^i)$ , then  $f(y_s) + 5 = f(x_{s+1}^i)$ , excluded by (3). If, by contradiction,  $g(x_s^{n_s-1}) = g(x_s^{n_s})$  then  $f(x_s^{n_s-1}) + 1 = f(y_s) + 5$ , so  $f(x_s^{n_s-1}) = f(y_s) + 4 = f(x_{s+1}^4)$  (by (3)) excluded as  $f$  is injective.
3.  $\max_A g = \max_A f + 1 < \min_B f + 1 = \min_B g$ , so  $g$  results in an  $\alpha_S$ -labeling of  $\Gamma$ .

**Remark 3.3.** *Obviously, we can apply more than one of the previous operations to the same standard  $\alpha_S$ -labeling, as long as they operate on disjoint set of edges.*

### 3.1 Hairy cycles

In this subsection we focus our attention on hairy cycles. In [1] Barrientos gives a labeling for any hairy cycle and when the graph is bipartite such a labeling is an  $\alpha$ -labeling.

Here we show that any bipartite hairy cycle  $H$  admits an odd  $\alpha$ -labeling (namely an  $e$ -divisible  $\alpha$ -labeling,  $e$  being the size of  $H$ ). Then, for any admissible value of  $d$ , we will prove the existence of  $d$ -divisible  $\alpha$ -labelings of an infinite class of hairy cycles, the coronas  $C_{2t} \odot \lambda K_1$ , see [10].

**Remark 3.4.** *Let  $H = HC(n_1, m_1, \dots, n_t, m_t)$  with  $t \geq 2$  be a bipartite hairy cycle. We will see  $H$  as the hairy cycle obtained from the caterpillar  $\Gamma$  of Figure 1 adding the edge  $[x_1, y_t]$ . Clearly, the number of edges of  $H$  is  $e = 2t + \sum_{i=1}^t n_i + \sum_{i=1}^t m_i$ . Let  $k = t + \sum_{i=1}^t n_i$ , that is  $k = |B|$ . Let  $e = d \cdot m$ ,  $\Delta = \{1, 2, \dots, e + d - 1\}$ ,  $\Delta' = \{m + 1, 2(m + 1), \dots, (d - 1)(m + 1)\}$ ,  $S = (\Delta \setminus \Delta') \setminus \{c\}$  where  $c \in \Delta \setminus \Delta'$ . Let  $f$  be the standard  $\alpha_S$ -labeling of  $\Gamma$ . The missing integers of  $\Delta'$  and the removal of  $c$  cause  $d$  mv-labels in  $f(A) \cup f(B)$ , let  $d_A$  and  $d_B$  be the number of mv-labels in  $f(A)$  and  $f(B)$ , respectively. Now  $f(y_t) - f(x_1) = f(y_t) - (f(x_1) + 1) + 1 = |B| + d_B = k + d_B$ . Thus if  $k + d_B = c$ , in a natural way we can extend  $f$  to a  $d$ -divisible  $\alpha$ -labeling of  $H$ , labeling the edge  $[x_1, y_t]$  by  $c$ .*

### 3.1.1 Odd (e-divisible) $\alpha$ -labelings of hairy cycles

**Theorem 3.5.** *A hairy cycle admits an odd  $\alpha$ -labeling if and only if it is bipartite.*

Proof. It is known that a hairy cycle with an odd  $\alpha$ -labeling is necessarily bipartite.

Suppose now  $H$  to be a bipartite hairy cycle. We start from the above Remark 3.4 and consider that  $e = d \cdot m$ ,  $d = e$  so  $m = 1$ , hence  $\Delta \setminus \Delta' = \{1, 3, 5, \dots, 2e-1\}$ . Thus, if the removal of  $c$  causes an extra mv-label in  $f(A)$  we will have  $d_B = k-1$  and  $f(y_t) - f(x_1) = 2k-1$ ; if the removal of  $c$  causes an extra mv-label in  $f(B)$  we will have  $d_B = k+1$  and  $f(y_t) - f(x_1) = 2k+1$ . Therefore, it will be convenient to choose  $c \in \{2k-1, 2k+1\}$  (note that this choice is always possible since  $\{2k-1, 2k+1\} \subseteq \Delta \setminus \Delta'$ ). In other words, the removal of  $c$  does not affect the values of the labels up to the  $(k-1)$ -th edge, the  $k$ -th edge will be labeled by  $2k-1$  if we choose  $c = 2k+1$  and vice versa, and after the labeling continues in a natural way up to the end. There are four possible cases.

- Case (1) If the  $k$ -th edge is a pendant edge from a vertex in  $B$ , the extra mv-label caused by the removal of  $c$  will be always in  $f(A)$ , thus  $f(y_t) - f(x_1) = 2k-1$ . Hence, choosing  $c = 2k-1$  and using  $2k+1$  as label of the  $k$ -th edge, we obtain an odd  $\alpha$ -labeling of  $H$ .
- Case (2) If the  $k$ -th edge is a pendant edge from a vertex in  $A$ , the extra mv-label caused by the removal of  $c$  will be always in  $f(B)$ , thus  $f(y_t) - f(x_1) = 2k+1$ . Hence, choosing  $c = 2k+1$  and using  $2k-1$  as label of the  $k$ -th edge, we obtain an odd  $\alpha$ -labeling of  $H$ .
- Case (3) If the  $k$ -th edge is an edge of the cycle of the form  $[x_s, y_{s-1}]$ ,  $c = 2k-1$  implies that the extra mv-label will be in  $f(A)$ , so  $f(y_t) - f(x_1) = 2k-1$ , while  $c = 2k+1$  implies that the extra mv-label will be in  $f(B)$ , so  $f(y_t) - f(x_1) = 2k+1$ . In both cases, the choice of the value of  $c$  results in appropriate.
- Case (4) If the  $k$ -th edge is an edge of the cycle of the form  $[x_s, y_s]$ ,  $c = 2k-1$  implies that the extra mv-label will be in  $f(B)$ , so  $f(y_t) - f(x_1) = 2k+1$  while  $c = 2k+1$  implies that the extra mv-label will be in  $f(A)$ , so  $f(y_t) - f(x_1) = 2k-1$ . So, no choice results in appropriate because the value we need to label  $[x_1, y_t]$  has been already used. To solve the problem, we have to distinguish two subcases: case (4<sub>1</sub>)  $n_{s+1} \neq 0$ , and case (4<sub>2</sub>)  $n_t \neq 0 \wedge m_t = 0$ . If we are able to define an odd  $\alpha$ -labeling in both previous subcases, we can do so in anycase. In fact, if the sequence  $(n_1, m_1, \dots, n_t, m_t)$  does not contains zeros, we refer to the Case (4<sub>1</sub>). If the sequence  $(n_1, m_1, \dots, n_t, m_t)$  contains at least one zero we refer to the Case (4<sub>2</sub>) because we can always choose another representation of  $H$  so that it results in  $n_t \neq 0 \wedge m_t = 0$ ,  $H$  being not a cycle.

Case (4<sub>1</sub>) We choose  $c = 2k-1$  and applying  $[O_2]$  we obtain another  $\alpha_S$ -labeling  $g$  with  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2 = 2k+1 - 2 = 2k-1$ .  $[O_2]$

can be used because both (1) and (2) of its definition hold, in fact: we know that  $f(y_s) - f(x_s) = 2k + 1$  and  $c = 2k - 1$  has been removed, so  $f(x_s^{n_s}) - f(x_s) = 2k - 3$ . Hence  $f(x_s^{n_s}) + 4 = f(y_s)$  and (2) holds. Moreover, from  $n_{s+1} \neq 0$  we have  $f(x_{s+1}^1) - f(x_{s+1}) = f(y_s) - f(x_{s+1}) + 2$ . Hence  $f(x_{s+1}^1) = f(y_s) + 2$  and (1) holds.

Case (4<sub>2</sub>) We choose  $c = 2k - 1$  and applying  $[O_4]$  we obtain another  $\alpha_S$ -labeling  $g$  with  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2 = 2k - 1$ .  $[O_4]$  can be used because both (1) and (2) of its definition hold by the hypothesis.  $\square$

For an explicit definition of the odd  $\alpha$ -labeling described in the above theorem see the Appendix.

**Example 3.6.** In Figure 4(a) we have the graph  $H = HC(3, 3, 0, 0, 3, 6, 0, 1, 3, 1)$ . Clearly we have  $e = 30$  and  $k = 14$ . In particular, the  $k$ -th edge is  $[x_3, y_3]$ . So, following the notation of the proof of Theorem 3.5 Case (4), we have  $s = 3$ . Since  $n_{s+1} = n_4 = 0$  and  $m_5 \neq 0$ , we have to rearrange the representation of  $H$  as  $HC(1, 3, 1, 3, 3, 0, 0, 3, 6, 0)$ . It just so happens that the  $k$ -th edge is still  $[x_3, y_3]$ , but now we are in the hypotheses of the case (4<sub>2</sub>), see Figure 4(b). The odd  $\alpha$ -labeling of  $H$  constructed following the proof of the previous theorem is shown in Figure 4(c).

Obviously, if after the rearrangement the  $k$ -th edge is not of the form  $[x_s, y_s]$  we are in one of the other cases and we apply the corresponding construction.

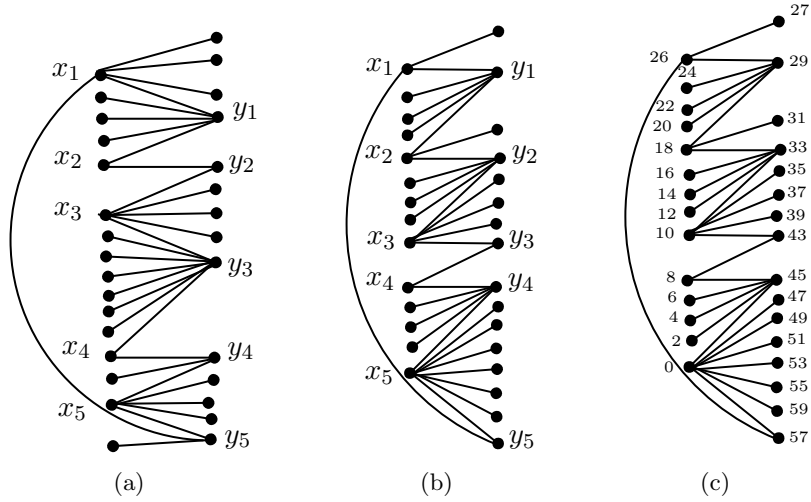


Figure 4:

As a consequence of Theorems 1.4 and 3.5 we have:

**Theorem 3.7.** Let  $\Gamma$  be a bipartite hairy cycle of size  $e$ . There exists a cyclic  $\Gamma$ -decomposition of  $K_{2 \times 2en}$  for any positive integer  $n$ .

### 3.1.2 $d$ -divisible $\alpha$ -labelings of $C_{2t} \odot \lambda K_1$

For convenience we will denote by  $H(2t, \lambda)$  the hairy cycle  $HC(\lambda, \lambda, \dots, \lambda)$  with the cycle of length  $2t$ ,  $t \geq 2$ , and with  $\lambda$ ,  $\lambda \geq 1$ , pendant edges through each vertex of the cycle. From [10] we learn that such a graph is nothing but the *corona* of  $C_{2t}$  with  $\lambda K_1$ , denoted by  $C_{2t} \odot \lambda K_1$ . Obviously  $H(2t, \lambda)$  has  $2t(\lambda + 1)$  edges.

**Theorem 3.8.** *The hairy cycle  $H(2t, \lambda)$  admits a  $d$ -divisible  $\alpha$ -labeling for any admissible value of  $d$ .*

*Proof.* If  $d = e$  the result follows from Theorem 3.5. So, from now on, we can assume  $d \neq e$ . Let  $\Gamma$  be the caterpillar obtained deleting the edge  $[x_1, y_t]$  from  $H(2t, \lambda)$  and let  $P$  be the path associated to  $\Gamma$ . We start again from the Remark 3.4 and we notice that now  $e = d \cdot m = 2t(\lambda + 1)$ ,  $k = e/2$  and the  $k$ -th edge is always an edge of the path  $P$ : of the form  $[x_r, y_r]$  if  $t$  is odd, of the form  $[x_r, y_{r-1}]$  if  $t$  is even.

Also, if  $d$  is odd the mv-labels due to the  $d - 1$  missing elements of  $\Delta'$  are equally distributed in  $f(A)$  and  $f(B)$ , because of the symmetry of the graph. So, if the removal of  $c$  causes an extra mv-label in  $f(A)$ , we will have  $d_B = \frac{d-1}{2}$  and  $f(y_t) - f(x_1) = \frac{e+d-1}{2}$ . If the removal of  $c$  causes an extra mv-label in  $f(B)$ , we will have  $d_B = \frac{d+1}{2}$  and  $f(y_t) - f(x_1) = \frac{e+d+1}{2}$ . Thus, it will be convenient to choose  $c \in \{c_1, c_2\}$  where  $c_1 = \frac{e+d-1}{2}$  and  $c_2 = \frac{e+d+1}{2}$ . Notice that  $\{c_1, c_2\} \subseteq \Delta \setminus \Delta'$  as if by contradiction  $\frac{e+d\pm 1}{2} = \mu(m+1)$  then  $(d-2\mu)(m+1) = \pm 1$ , that is impossible.

If  $d$  is even, because of the symmetry of the graph, the mv-labels due to the  $d - 2$  elements of  $\Delta' \setminus \{e/2\}$  are equally distributed in  $f(A)$  and  $f(B)$ . So, if the removal of  $c$  causes the two further mv-labels in  $f(A)$ , we will have  $d_B = \frac{d-2}{2}$  and  $f(y_t) - f(x_1) = \frac{e+d-2}{2}$  as well as if the removal of  $c$  causes the two further mv-labels in  $f(B)$ , we will have  $d_B = \frac{d+2}{2}$  and  $f(y_t) - f(x_1) = \frac{e+d+2}{2}$ . Thus, it will be convenient to choose  $c \in \{c_1, c_2\}$  where  $c_1 = \frac{e+d-2}{2}$  and  $c_2 = \frac{e+d+2}{2}$ . Notice that  $\{c_1, c_2\} \subseteq \Delta \setminus \Delta'$  since  $\frac{e+d\pm 2}{2} = \frac{d}{2}(m+1) \pm 1 \notin \Delta'$ .

In other words, in both cases the removal of  $c$  does not affect the values of the labels up to the  $(k-1)$ -th edge, the  $k$ -th edge will be labelled by  $c_1$  if we have chosen  $c = c_2$  and vice versa, and after the labeling continues, in ascending order, up to the end. We continue by distinguishing several cases.

- Case (1) Let  $t$  be even, that is the  $k$ -th edge is of the form  $[x_r, y_{r-1}]$  with  $r = \frac{t+2}{2}$ .

Let  $S = (\Delta \setminus \Delta') \setminus \{c_2\}$ . Let  $f$  be the standard  $\alpha_S$ -labeling of  $\Gamma$ . The choice of  $c = c_2$  implies that the extra mv-labels are always in  $f(B)$ , so  $f(y_t) - f(x_1) = c_2$  and, in a natural way,  $f$  can be extended to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

- Case (2) Let  $t$  be odd, that is the  $k$ -th edge is of the form  $[x_r, y_r]$  with  $r = \frac{t+1}{2}$ .

Let  $S = (\Delta \setminus \Delta') \setminus \{c\}$  where  $c$  can be chosen in  $\{c_1, c_2\}$ . Let  $f$  be the standard  $\alpha_S$ -labeling of  $\Gamma$ . The choice  $c = c_1$  implies that the extra mv-labels are always in  $f(B)$ , so  $f(y_t) - f(x_1) = c_2$ . The choice  $c = c_2$  implies that the extra mv-labels are always in  $f(A)$ , so  $f(y_t) - f(x_1) = c_1$ . In both cases the value we need is not available. We can proceed by distinguishing two subcases.

Case (2<sub>1</sub>) Let  $d$  be odd. Now  $\{c_1, c_2\} = \{\frac{e+d-1}{2}, \frac{e+d+1}{2}\}$ . We choose  $c = c_1$ , consider the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$  and apply  $[O_1]$  to  $f$  with  $s = \frac{t+1}{2}$ . This is possible because (1) and (2) are satisfied, in fact now we have  $f(y_s) - f(x_s) = c_2$  and  $f(x_s^{n_s}) - f(x_s) = c_1 - 1$  if  $c_1 - 1 \notin \Delta'$  or  $f(x_s^{n_s}) - f(x_s) = c_1 - 2$  if  $c_1 - 1 \in \Delta'$ , so  $f(y_s) - f(x_s^{n_s}) = 2$  or  $3$  and (2) is true. Also, if (1) was not true, there would be a mv-label between  $f(y_s)$  and  $f(x_{s+1}^1)$  and, symmetrically, there would be a mv-label between  $f(x_s)$  and  $f(y_{s-1}^{m_{s-1}})$ . Thus  $m$  should be a divisor of  $2\lambda + 3$ , an odd number, while  $m$  is obviously even as now  $d$  is odd. Applying  $[O_1]$  we obtain an  $\alpha_S$ -labeling  $g$  of  $\Gamma$  with  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1 = c_2 - 1 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

Case (2<sub>2</sub>) Let  $d$  be even. Now  $\{c_1, c_2\} = \{\frac{e+d-2}{2}, \frac{e+d+2}{2}\}$ . We have to split the proof in several subcases.

Case (2<sub>2,1</sub>) Let  $\lambda \geq 2$  and  $\lambda \not\equiv m - 2, m - 3 \pmod{m}$ .

Choose  $c = c_1$  and let  $f$  be the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , apply  $[O_3]$  to  $f$  with  $s = \frac{t+1}{2}$ . This is possible because (1), (2) and (3) are satisfied, in fact now we have  $f(y_s) - f(x_s) = c_2$  and  $f(x_s^{n_s}) - f(x_s) = c_1 - 1$  if  $c_1 - 1 \notin \Delta'$  or  $f(x_s^{n_s}) - f(x_s) = c_1 - 2$  if  $c_1 - 1 \in \Delta'$  so  $f(y_s) - f(x_s^{n_s}) = 3$  or  $4$  and (2) is true. Also, if (1) was not true, there would be a mv-label either between  $f(y_s)$  and  $f(x_{s+1}^1)$  (excluded as  $\lambda \not\equiv m - 2 \pmod{m}$ ) or between  $f(x_{s+1}^1)$  and  $f(x_{s+1}^2)$  (excluded as  $\lambda \not\equiv m - 3 \pmod{m}$ ). Applying  $[O_3]$  we obtain an  $\alpha_S$ -labeling  $g$  of  $\Gamma$  with  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2 = c_2 - 2 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

Case (2<sub>2,2</sub>) Let  $\lambda \geq 2$  and  $\lambda \equiv m - 2 \pmod{m}$ .

Choose  $c = c_1$  and let  $f$  be the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , apply  $[O_2]$  to  $f$  with  $s = \frac{t+1}{2}$ . This is possible because (1) and (2) are satisfied, in fact now we have  $f(y_s) - f(x_s) = c_2$  and  $f(x_s^{n_s}) - f(x_s) = c_1 - 1$  if  $c_1 - 1 \notin \Delta'$  or  $f(x_s^{n_s}) - f(x_s) = c_1 - 2$  if  $c_1 - 1 \in \Delta'$  so  $f(y_s) - f(x_s^{n_s}) = 3$  or  $4$  and (2) is true. Also, there is a mv-label between  $f(y_s)$  and  $f(x_{s+1}^1)$  as  $\lambda \equiv m - 2 \pmod{m}$ , so  $f(y_s) - f(x_{s+1}^1) + 2 = f(x_{s+1}^1) - f(x_{s+1}^2)$  and (1) is true. Applying  $[O_2]$  we obtain an  $\alpha_S$ -labeling  $g$  of  $\Gamma$  with  $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2 = c_2 - 2 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

Case (2<sub>2,3</sub>) Let  $\lambda \equiv m - 3 \pmod{m}$  and

- $\lambda \geq 4 \wedge m \geq 4 \wedge t \geq 5$  or  $\lambda \geq 7 \wedge m \geq 4 \wedge t = 3$ .

Choose  $c = c_1$ , consider the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , say  $f$ , and apply  $[O_1]$  and  $[O_5]_4$  to obtain an  $\alpha_S$ -labeling  $g$  of  $\Gamma$ . We can apply  $[O_1]$  because  $\lambda \equiv m - 3 \pmod{m}$ , and we can apply  $[O_5]_4$  as  $\lambda \geq 4 \wedge m \geq 4$ . We can apply both  $[O_1]$  and  $[O_5]_4$ , in any order, as  $t \geq 5$  or  $t = 3 \wedge 7 \leq \lambda$ , and this ensures that there is no edge affected by both the operations. Thus we have  $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

- $\lambda \geq 4 \wedge m \geq 4 \wedge t = 3 \wedge \lambda < 7$ .

From our hypotheses we have

$4 \leq \lambda < 7$	$\lambda + 3$	$e = 6(\lambda + 1)$	$m \geq 4$ divides $\lambda + 3$ and $e$
4	7	30	there is no value
5	8	36	4 unacceptable, as $d$ is even
6	9	42	there is no value

- $m = 2, t \geq 3$ .

Choose  $c = c_1$ , consider the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , say  $f$ , and apply  $[O_1]$  with  $s = \frac{t+1}{2}$  and  $[O_5]$  with  $s = 1$  to obtain an  $\alpha_S$ -labeling  $g$  of  $\Gamma$ . We can apply  $[O_1]$  because  $\lambda \equiv m - 3 \pmod{m}$ , and we can apply  $[O_5]$  as  $m$  divides  $2(\lambda + 1)$ . We can apply both  $[O_1]$  and  $[O_5]$ , in any order, as  $\lambda \geq 2$  and this ensures that there is no edge affected by both the operations. Thus we have  $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$ . So we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

- $m = 3, t \geq 3$ .

Choose  $c = c_1$  and consider the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , say  $f$ . We can apply  $[O_1]$  with  $s = \frac{t+1}{2}$  because  $\lambda \equiv m - 3 \pmod{m}$ , and we can apply  $[O_1]$  with  $s = 1$  as  $m$  divides  $\lambda$ . We can apply twice  $[O_1]$  with  $s = 1$  and  $s = \frac{t+1}{2}$ , in any order, as  $\lambda \geq 2$  and this ensures that there is no edge affected by both the operations.

Thus we have  $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

- $m \geq 4 \wedge \lambda = 2$ .

It happens only when  $m = 5$  e  $\lambda = 2$ . Then  $e = 6t$  implies that  $t$  have to be an odd multiple of 5. We choose  $c = c_1$  and consider the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , say  $f$ . It easy to see that we can apply both  $[O_1]$  with  $s = \frac{t+1}{2}$  and  $[O_5]$  with  $s = \frac{t+3}{2}$ , in any order, being sure that there is no edge affected by both the operations.

Thus we have  $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

- $m \geq 4 \wedge \lambda = 3$ .

It happens only when  $m = 6$  e  $\lambda = 3$ . Then  $e = 8t$  implies that  $t$  have to be an odd multiple of 3. If  $t = 3$ , we are able to give directly a  $d$ -divisible  $\alpha$ -

labeling of  $H(2t, \lambda)$ . If  $t \geq 5$ , we choose  $c = c_1$  and consider the corresponding standard  $\alpha_S$ -labeling of  $\Gamma$ , say  $f$ . It is easy to see that we can apply both  $[O_1]$  with  $s = \frac{t+1}{2}$  and  $[O_5]$  with  $s = 3$ , in any order, being sure that there is no edge affected by both the operations.

Thus we have  $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$ . So, we can extend  $g$  to a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ , labeling the edge  $[x_1, y_t]$  by  $c_1$ .

If  $t = 3$ , we are able to give directly a  $d$ -divisible  $\alpha$ -labeling of  $H(2t, \lambda)$ .

Case  $(2_{2_4})$  Let  $\lambda = 1$ . Also in this case the foregoing construction can be applied, by distinguishing the cases  $m = 0, 1, 2, 3 \pmod{4}$  and applying in a suitable way the  $[O_i]$ 's. Here, in Appendix, is given an explicit 2-divisible  $\alpha$ -labeling of  $H(2t, 1)$  for any  $t$  odd.  $\square$

**Example 3.9.** Here we show a 6-divisible  $\alpha$ -labeling of the hairy cycle  $HC(10, 2)$ . Since  $t = 5$ ,  $d = 6$ ,  $m = 5$  and  $\lambda = 2$ , we are in the Case  $(2_{2_3})$  of Theorem 3.8. In Figure 5 (a) we have the standard  $\alpha_S$ -labeling  $f$  of the caterpillar  $\Gamma$  obtained by  $HC(10, 2)$  deleting the edge  $[x_1, y_5]$  with  $S = (\Delta \setminus \Delta') \setminus \{17\}$  where  $\Delta = \{1, 2, \dots, 35\}$  and  $\Delta' = \{6, 12, 18, 24, 30\}$ . Now  $f(y_5) - f(x_1) = 19$  which is not available, so we cannot extend  $f$  to a 6-divisible  $\alpha$ -labeling of  $HC(10, 2)$ . It is easy to see that it is possible to apply  $[O_1]$  with  $s = 3$ , the  $\alpha_S$ -labeling  $g$  so obtained is shown in Figure 5 (b). Now  $g(y_5) - g(x_1) = 18$  which is again not available as  $18 \in \Delta'$ . Then we can apply  $[O_5]$  with  $s = 4$ . After this we obtain the  $\alpha_S$ -labeling  $\tilde{g}$  of  $\Gamma$  shown in Figure 5 (c). Finally  $\tilde{g}(y_5) - \tilde{g}(x_1) = 17$ . Now we can extend  $\tilde{g}$  to a 6-divisible  $\alpha$ -labeling of  $HC(10, 2)$ , labeling the edge  $[x_1, y_5]$  by 17.

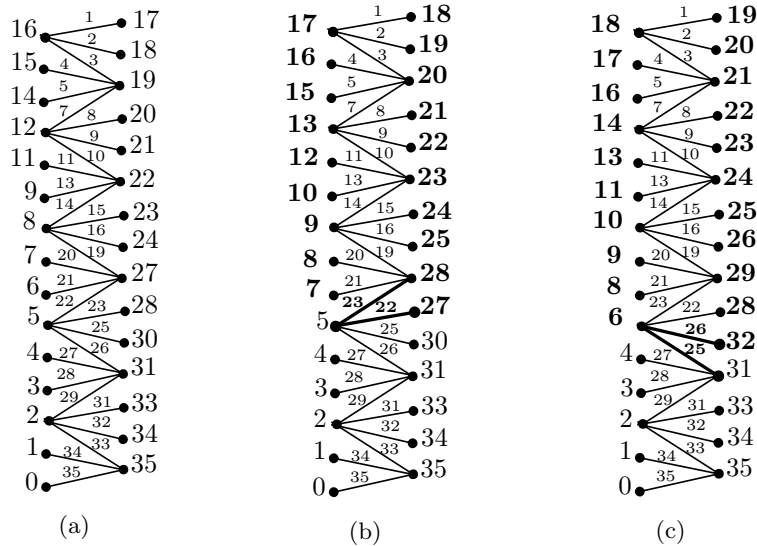


Figure 5:



The following is an immediate consequence of Theorems 1.4 and 3.8.

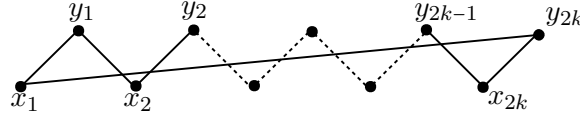
**Theorem 3.10.** *There exists a cyclic  $H(2t, \lambda)$ -decomposition of  $K_{\left(\frac{2t(\lambda+1)}{d}+1\right) \times 2dn}$  for any admissible  $d$  and for any positive integer  $n$ .*

### 3.2 Cycles

As usual, we will denote the cycle on  $k$  vertices by  $C_k$ ,  $k \geq 3$ . It is obvious that  $C_k$  is a graph of size  $k$  and that it is bipartite if and only if  $k$  is even. In [16] Rosa proved that  $C_k$  has an  $\alpha$ -labeling if and only if  $k \equiv 0 \pmod{4}$ . In [15], the second author proved that  $C_{4k}$  admits a 2-divisible and a 4-divisible  $\alpha$ -labeling for any positive integer  $k$ . Here, generalizing this last result, we prove that  $C_{4k}$  admits a  $d$ -divisible  $\alpha$ -labeling for any divisor  $d$  of  $4k$ .

**Theorem 3.11.** *For any positive integer  $k$ , the cycle  $C_{4k}$  admits a  $d$ -divisible  $\alpha$ -labeling for any admissible value of  $d$ .*

Proof. Consider the cycle  $C_{4k}$  as a bipartite graph as follows:



and set  $A = \{x_1, x_2, \dots, x_{2k}\}$  and  $B = \{y_1, y_2, \dots, y_{2k}\}$  the two bipartite sets. Let  $\Gamma$  be the caterpillar obtained from  $C_{4k}$  deleting the edge  $[x_1, y_{2k}]$ . Let  $4k = d \cdot m$ ,  $\Delta = \{1, 2, \dots, 4k+d-1\}$  and  $\Delta' = \{m+1, 2(m+1), \dots, (d-1)(m+1)\}$ . Choose an element in  $\Delta \setminus \Delta'$ , say  $c$ . Let  $f$  be the standard  $\alpha_S$ -labeling of  $\Gamma$  where  $S = (\Delta \setminus \Delta') \setminus \{c\}$ . In order to show that  $f$  can be naturally extended to a  $d$ -divisible  $\alpha$ -labeling of  $C_{4k}$  it remains to prove that we can choose the element  $c$  so that  $c = f(y_{2k}) - f(x_1)$ , see Remark 3.2.

Let  $d_x$  and  $d_y$  denote the numbers of the mv-labels in  $f(A)$  and  $f(B)$ , respectively. Obviously, the condition  $c = f(y_{2k}) - f(x_1)$  becomes  $c = |B| + d_y = 2k + d_y$ . In what follows, we are able to determine  $d_y$  and, consequently,  $c$ . If  $m = 4k/d$  is even, the elements of  $\Delta'$  are odd and even, alternatively, so  $c$  lies between two elements of different parity. The deletion of any element of  $\Delta'$  smaller than  $c$  produces a mv-label in  $f(B)$ , while deleting an element of  $\Delta'$  greater than  $c$  gives a mv-label in  $f(A)$ . Thus we have necessarily  $d_y(m+1) < c < (d_y+1)(m+1)$ , where  $c = d_y + 2k$  and  $2k = md/2$ . With a simple calculation we obtain  $d_y m < md/2 \leq (d_y+1)m$ , from which  $2d_y < d \leq 2d_y + 2$ . So,  $d$  even implies  $d_y = (d-2)/2$  as well as  $d$  odd implies  $d_y = (d-1)/2$ .

If  $m = 4k/d$  is odd, all the elements of  $\Delta'$  are even. If  $c$  was even too, we would have  $d_x = d_y = d/2$ , so  $c = 2k + d/2 = d/2(m+1) \in \Delta'$ , but  $c$  must not belong to  $\Delta'$ . Thus  $c$  must be odd and we have  $d_x - d_y = \pm 2$ . In addition,

we know that  $d_x + d_y = d$ , so  $d_y = (d \mp 2)/2$ .  $\square$

**Remark 3.12.** *It is known that if  $f$  is a ( $d$ -divisible)  $\alpha$ -labeling of a bipartite graph  $\Gamma$  of size  $e$ , the function  $g : V(\Gamma) \rightarrow \{0, 1, \dots, e\}$ , defined by  $g(x) = e - f(x)$ ,  $\forall x \in V(\Gamma)$  is again a ( $d$ -divisible)  $\alpha$ -labeling of  $\Gamma$ . We point out that if  $d = 1$ , called  $f$  the  $\alpha$ -labeling constructed in the above theorem and  $g$  the classical  $\alpha$ -labeling given by A. Rosa in [16] it results  $g(x) = e - f(x)$ ,  $\forall x \in V(\Gamma)$ . Also, when  $d = 2, 4$  the same relation holds between the  $d$ -divisible  $\alpha$ -labeling constructed in the above theorem and that given by the second author in [15].*

**Example 3.13.** *In Figure 6 we show the  $d$ -divisible  $\alpha$ -labelings of  $C_{24}$  described in Theorem 3.11 for  $d = 3$  and  $d = 8$ .*

*If  $d = 3$ , we have  $m + 1 = 9$  and  $c = 13$ . If  $d = 8$  it results  $m + 1 = 4$  and we can choose  $c = 15$  or  $c = 17$  (as in Figure 6).*

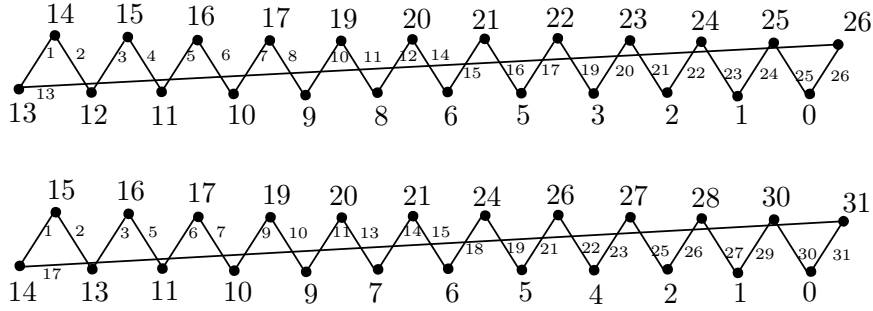


Figure 6:

The following result immediately follows from Theorems 1.4 and 3.11.

**Theorem 3.14.** *There exists a cyclic  $C_{4k}$ -decomposition of  $K_{(\frac{4k}{d}+1) \times 2dn}$  for any positive integers  $k, n$  and any divisor  $d$  of  $4k$ .*

## Appendix

Here we give an explicit representation of the odd  $\alpha$ -labeling  $f$  of a bipartite hairy cycle  $\Gamma$  of size  $e$  constructed in Theorem 3.5. We give the definition of the function  $f : V(\Gamma) \rightarrow \{0, 1, \dots, 2e - 1\}$  in each of the possible cases.

Case (1): the  $k$ -th edge is a pendant edge from a vertex in  $B$ , say  $[y_s, y_s^j]$ .

$$f(x_r) = \begin{cases} 2 \sum_{\ell=r}^t m_\ell + 2(t-r) + 2 & r = 1, \dots, s \\ 2 \sum_{\ell=r}^t m_\ell + 2(t-r) & r = s+1, \dots, t \end{cases}$$

$$f(y_r) = f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r - 1 \quad r = 1, \dots, t$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1 & r = 1, i = 1, \dots, n_1 \\ f(y_{r-1}) + 2i & r = 2, \dots, t, i = 1, \dots, n_r \end{cases}$$

$$f(y_r^i) = \begin{cases} f(x_r) - 2i & r \neq s, i = 1, \dots, m_r \\ f(x_r) - 2(i+1) & r = s, i = 1, \dots, j-1 \\ f(x_r) - 2(i+1) & r = s, i = j, \dots, m_s. \end{cases}$$

Case (2): the  $k$ -th edge is a pendant edge from a vertex in  $A$ , say  $[x_s, x_s^j]$ .

$$f(x_r) = 2 \sum_{\ell=r}^t m_\ell + 2(t-r) \quad r = 1, \dots, t$$

$$f(y_r^i) = f(x_r) - 2i \quad r = 1, \dots, t, i = 1, \dots, m_r$$

$$f(y_r) = \begin{cases} f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r - 1 & r = 1, \dots, s-1 \\ f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r + 1 & r = s, \dots, t \end{cases}$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1 & r = 1, i = 1, \dots, n_1 \\ f(y_{r-1}) + 2i & r \neq 1, s, i = 1, \dots, n_r \\ f(y_{r-1}) + 2i & r = s, i = 1, \dots, j \\ f(y_{r-1}) + 2i + 2 & r = s, i = j+1, \dots, n_s, \text{ if } j \neq n_s. \end{cases}$$

Case (3): the  $k$ -th edge is an edge of the cycle of the form  $[x_s, y_{s-1}]$ .

$$f(x_r) = \begin{cases} 2 \sum_{\ell=r}^t m_\ell + 2(t-r) + 2 & r = 1, \dots, s-1 \\ 2 \sum_{\ell=r}^t m_\ell + 2(t-r) & r = s, \dots, t \end{cases}$$

$$f(y_r) = f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r - 1 \quad r = 1, \dots, t$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1 & r = 1, i = 1, \dots, n_1 \\ f(y_{r-1}) + 2i & r = 2, \dots, t, i = 1, \dots, n_r \end{cases}$$

$$f(y_r^i) = f(x_r) - 2i \quad r = 1, \dots, t, i = 1, \dots, m_r.$$

Case (4): the  $k$ -th edge is an edge of the cycle of the form  $[x_s, y_s]$ .

Case (4<sub>1</sub>):  $n_{s+1} \neq 0$ .

$$f(x_r) = \begin{cases} 2 \sum_{\ell=r}^t m_\ell + 2(t-r) + 2 & r = 1, \dots, s \\ 2 \sum_{\ell=r}^t m_\ell + 2(t-r) & r = s+1, \dots, t \end{cases}$$

$$f(y_r^i) = f(x_r) - 2i \quad r = 1, \dots, t, i = 1, \dots, m_r$$

$$f(y_r) = \begin{cases} f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r - 1 & r \neq s \\ f(x_1) + 2 \sum_{\ell=1}^s n_\ell + 2s + 1 & r = s \end{cases}$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1 & r = 1, i = 1, \dots, n_1 \\ f(y_{r-1}) + 2i & r = 2, \dots, s, s+2, \dots, t, i = 1, \dots, n_r \\ f(y_s) - 2 & r = s+1, i = 1 \\ f(y_s) + 2(i-1) & r = s+1, i = 2, \dots, n_{s+1}. \end{cases}$$

Case (4<sub>2</sub>) :  $n_t \neq 0, m_t = 0$ .

$$f(x_r) = 2 \sum_{\ell=r}^t m_\ell + 2(t-r) \quad r = 1, \dots, t$$

$$f(y_r^i) = f(x_r) - 2i \quad r = 1, \dots, t-1, i = 1, \dots, m_r$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1 & r = 1, i = 1, \dots, n_1 \\ f(y_{r-1}) + 2i - 1 & r = 2, \dots, t-1, i = 1, \dots, n_r \\ & r = t, i = 1, \dots, n_t - 1 \\ 2e - 1 & r = t, i = n_t \end{cases}$$

$$f(y_r) = \begin{cases} f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r - 1 & r = 1, \dots, s-1 \\ f(x_1) + 2 \sum_{\ell=1}^r n_\ell + 2r + 1 & r = s, \dots, t-1 \\ 2e - 3 & r = t. \end{cases}$$

In the following we give an explicit construction of a 2-divisible  $\alpha$ -labeling of  $H(2t, 1)$  for  $1 < t$  odd, whose existence has been proved in Theorem 3.11. We have to distinguish two cases according to the congruence class of  $t$  modulo 4.

Case (1):  $t \equiv 1 \pmod{4}$

$$f(x_r) = \begin{cases} 2t+3-2r & r = 1, \dots, \frac{t+3}{4} \\ 2t+2-2r & r = \frac{t+7}{4}, \dots, \frac{t+1}{2} \\ 2t+1-2r & r = \frac{t+3}{2}, \dots, t \end{cases} \quad f(y_r^1) = \begin{cases} 2t+2-2r & r = 1, \dots, \frac{t-1}{4} \\ 2t+1-2r & r = \frac{t+3}{4}, \dots, \frac{t-1}{2} \\ 2t-2r & r = \frac{t+1}{2}, \dots, t \end{cases}$$

$$f(x_r^1) = \begin{cases} 3t+2 & r = 1 \\ 2t-1+2r & r = 2, \dots, \frac{t+1}{2} \\ 3t+1 & r = \frac{t+3}{2} \\ 2t+2r & r = \frac{t+5}{2}, \dots, t \end{cases} \quad f(y_r) = \begin{cases} 2t+2r & r = 1, \dots, \frac{t-1}{2} \\ 3t+3 & r = \frac{t+1}{2} \\ 2t+1+2r & r = \frac{t+3}{2}, \dots, t \end{cases}$$

Case (2):  $t \equiv 3 \pmod{4}$

$$f(x_r) = \begin{cases} 2t+3-2r & r = 1, \dots, \frac{t+1}{4} \\ 2t+2-2r & r = \frac{t+5}{4}, \dots, \frac{t+1}{2} \\ 2t+1-2r & r = \frac{t+3}{2}, \dots, t \end{cases} \quad f(y_r^1) = \begin{cases} 2t+2-2r & r = 1, \dots, \frac{t+1}{4} \\ 2t+1-2r & r = \frac{t+5}{4}, \dots, \frac{t+1}{2} \\ 2t-2r & r = \frac{t+3}{2}, \dots, t \end{cases}$$

$$f(x_r^1) = \begin{cases} 3t+1 & r=1 \\ 2t-1+2r & r=2, \dots, \frac{t+3}{2} \\ 2t+2r & r=\frac{t+5}{2}, \dots, t, \text{ if } t>3 \end{cases} \quad f(y_r) = \begin{cases} 2t+2r & r=1, \dots, \frac{t-1}{2} \\ 3t+3 & r=\frac{t+1}{2} \\ 2t+1+2r & r=\frac{t+3}{2}, \dots, t \end{cases}$$

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